The Minimax Inequality for Lattices

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The Minimax Inequality,

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \leqslant \inf_{y\in Y}\sup_{x\in X} f(x,y),$$

holds for any function $f : X \times Y \to \mathbb{R}$. There's a large assortment of theorems called the "Minimax Theorem" that give necessary and/or sufficient conditions for equality: the most famous one, due to John von Neumann, has an interpretation in terms of optimal strategies in game theory.

As an example where equality fails, let $f(j,i)$ be the i,j component of the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The left side of the inequality is the maximum of the column minima, and the right side is the minimum of row maxima, so the inequality holds strictly: $0 < 1$.

The Minimax Inequality is most naturally expressed in the language of lattice theory.

Theorem. *Let L be a lattice, and let X and Y be sets. For any indexed collection of elements $(a_{xy})_{x\in X,\, y\in Y}$ in L,*

$$\bigvee_{x\in X}\left(\bigwedge_{y\in Y} a_{xy}\right) \leqslant \bigwedge_{y\in Y}\left(\bigvee_{x\in X} a_{xy}\right),$$

provided all these joins and meets exist in L (as happens for example when X and Y are nonempty and finite, or when L is a complete lattice.)

Proof. For every $\bar{x} \in X$,

$$a_{\bar{x}y} \leqslant \bigvee_{x\in X} a_{xy}.$$

Take meets over Y to get

$$\bigwedge_{y\in Y} a_{\bar{x}y} \leqslant \bigwedge_{y\in Y}\left(\bigvee_{x\in X} a_{xy}\right),$$

and then take the join on the left side over all $\bar{x} \in X$. $\square$

Here's a (slightly silly) application: Let L be the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$, take $X = Y = \mathbb{N}_{>0}$, and let a_{xy} be the elements of the infinite Hankel matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $a_0, a_1, \ldots$ are extended reals. Then the Minimax Inequality reduces to

$$\liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n.$$

There are also a couple of algebraic consequences for lattices. First, when applied to the matrix

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix},$$

the inequality becomes the *distributive inequality*

$$(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c).$$

Second, when applied to

$$\begin{bmatrix} a & b & a \\ b & b & c \\ a & c & c \end{bmatrix},$$

it becomes the *median inequality*

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leqslant (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

1 References

The lattice-theoretic Minimax Inequality and its two algebraic consequences are from

- B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order (2nd Edition)*, 2002, p. 58.

For more about the game-theoretic setting, and for applications to optimization, see

- Dimitri P. Bertsekas, *Convex Optimization Theory*, 2009, pp. 127-130.
- Eberhard Zeidler, *Nonlinear Functional Analysis and its Applications, Volume I: Fixed-Point Theorems*, 1986, pp. 461-463.